Numerical Loop-Integration Methods for Finite Temperature Effects in QCD Sum Rules

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Thermal Field Theory

- A extension of ordinary Quantum Field Theory at zero temperature.
- There are interesting physical problems that are depart from this situation.
- For example, particles in the early universe, the electroweak phase transition [1], the quark-gluon plasma, gravitational wave signals from strong first-order phase transitions [2].

Matsubara Formalism

- The contour of the density operator is redefined analogous to the approach of Feynman rules at zero temperature is $\mathcal{C} = [t_i, +\infty] \cup [+\infty, t_i] \cup [t_i, t_i - i\beta]$. [3]

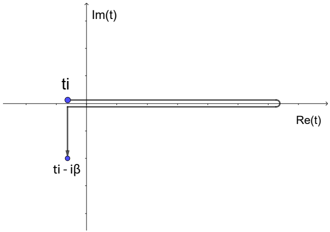


Figure: The integration contour $\mathcal{C}$ of thermal time for the density operator $e^{-\rho\mathcal{H}}$ in the complex time plane.

Matsubara Formalism

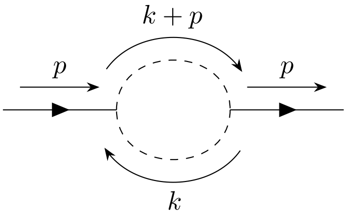
- The Feynman diagram integrand can be Fourier transformed into frequency space. For free bosonic scalar fields,

$$\mathcal{G}^0(\omega_n, \boldsymbol{p}) = \frac{1}{\omega_n^2 + \boldsymbol{p}^2 + m^2}$$

where the Matsubara frequencies are $\omega_n = 2\pi n \mathcal{T}$ for $\mathcal{T}$ being temperature. [3]

pySecDec Numerical Calculation

- pySecDec: a package designed for numerical calculation of dimensionally regulated loop integrals.
- The Feynman diagram topology representing the one-loop self-energy integral. [3]



$$\Pi(\boldsymbol{p}, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\frac{4n^2\pi^2}{\beta^2} + \boldsymbol{k}^2 + M^2} \times \frac{1}{(\frac{2n\pi}{\beta} + p^0)^2 + (\boldsymbol{k} + \boldsymbol{p})^2 + M^2}$$

pySecDec Euclideanization

$$\Pi(\boldsymbol{p}, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\boldsymbol{k}^2 + m_1^2} \times \frac{1}{(\boldsymbol{k}+\boldsymbol{p})^2 + m_2^2}$$

where we take $m_1^2 = \frac{4n^2\pi^2}{\beta^2} + M^2$ and $m_2^2 = (\frac{2n\pi}{\beta} + p^0)^2 + M^2$.

- Using pySecDec to do the d-dimensional spatial integration part, then write separate python codes for the discretized series of temporal sum.
- Momentum in correlation function is in Euclidean space whereas pySecDec works on loop integrals in Minkowski space.
- We need to do inverse Euclideanization to the input momentum(p_1) by applying Wick rotation.

Methodology for pySecDec Numerical Calculation

$$p_0^2 - p_1^2 - p_2^2 \tag{1}$$

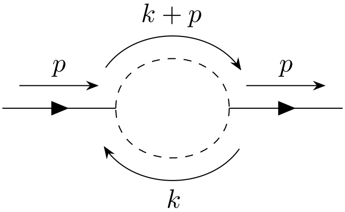
(pySecDec)

$$\downarrow$$

$$-p_1^2 - p_2^2 - p_3^2 \tag{2}$$

(what we want for pySecDec to do is pure spatial integration)

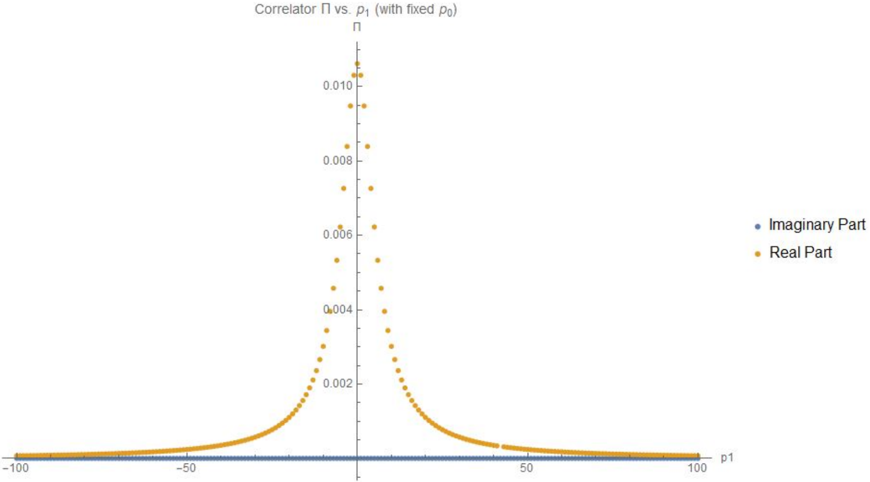
pySecDec Numerical Calculation: 1+1 Dimension



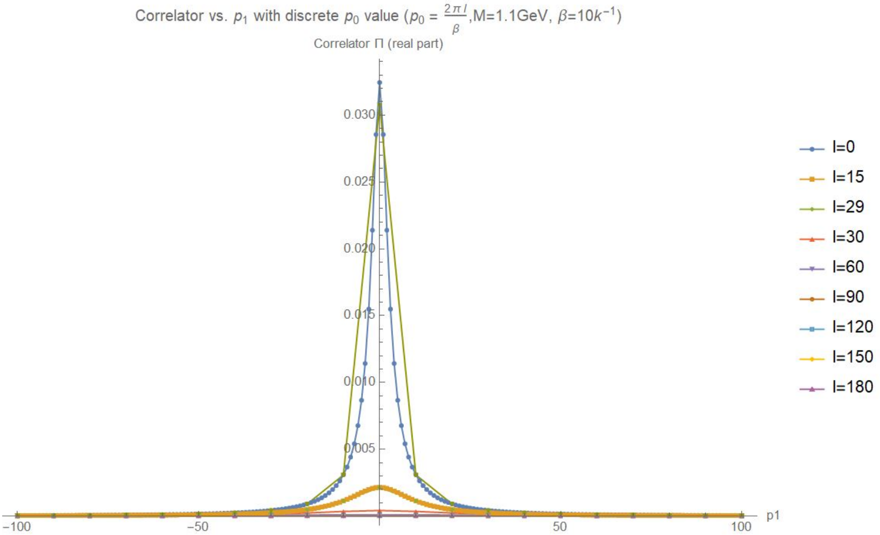
$$\Pi(p_1, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{dk_1}{2\pi} \frac{1}{\frac{4n^2\pi^2}{\beta^2} + k_1^2 + M^2} \times \frac{1}{(\frac{2n\pi}{\beta} + p^0)^2 + (k_1 + p_1)^2 + M^2}$$

where we take $m_1^2 = \frac{4n^2\pi^2}{\beta^2} + M^2$ and $m_2^2 = (\frac{2n\pi}{\beta} + p^0)^2 + M^2$.

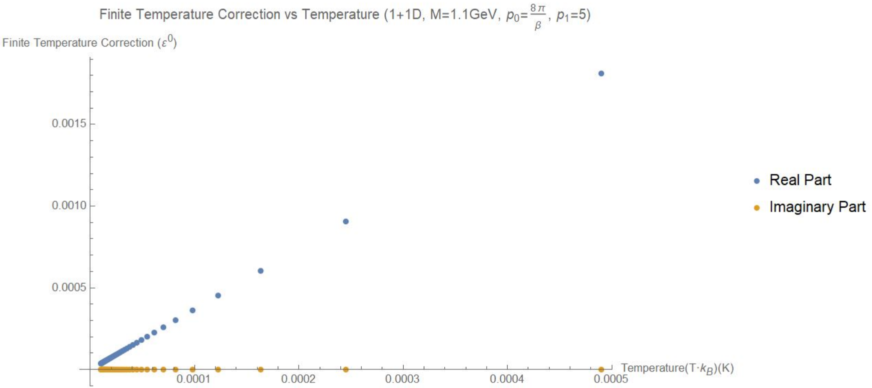
pySecDec Numerical Calculation: 1+1 Dimension



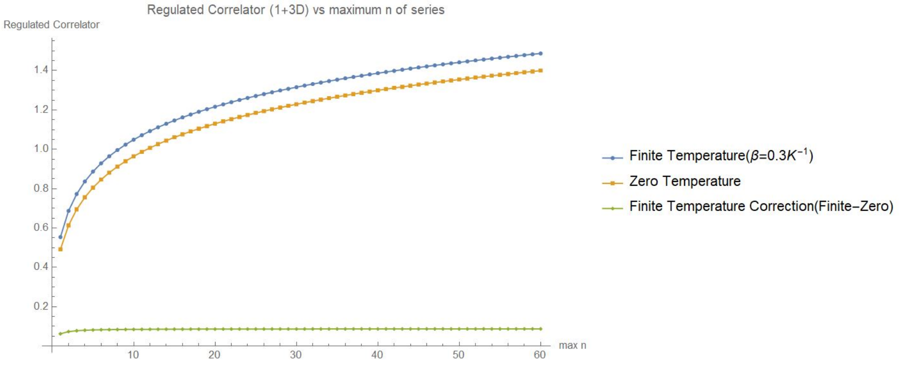
pySecDec Numerical Calculation: 1+1 Dimension



pySecDec Numerical Calculation: 1+1 Dimension



pySecDec Numerical Calculation: 1+3 Dimension



Summary

- We have developed a numerical calculation method for Thermal Field Theory focusing on the one-loop self-energy topology that appears in QCD Sum Rules.
- Our methodology incorporates pySecDec for calculating part of the Thermal Field Theory integration and additional coding for the discretized dimension.
- There are still many problems and questions emerging in calculations from a variety of dimensions.

Thank You!

References

[1] M. Gogberashvili, "Electroweak Phase Transitions in Einstein's Static Universe," *Adv. High Energy Phys.*, vol. 2018, p. 4653202, 2018.

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[3] A. Das, *Finite Temperature Field Theory*. World scientific lecture notes in physics, World Scientific, 1997.

[4] P. Colangelo and A. Khodjamirian, "QCD sum rules, a modern perspective," pp. 1495–1576, 10 2000.

[5] S. Borowka, G. Heinrich, S. Jahn, S. Jones, M. Kerner, J. Schlenk, and T. Zirke, "pySecDec: a toolbox for the numerical evaluation of multi-scale integrals," *Comput. Phys. Commun.*, vol. 222, pp. 313–326, 2018.

[6] A. I. Davydychev, "Some exact results for N point massive Feynman integrals," *J. Math. Phys.*, vol. 32, pp. 1052–1060, 1991.

[7] P. Pascual and R. Tarrach, *QCD: Renormalization for the Practitioner*. Lecture notes in physics, Springer-Verlag, 1984.

Backup Slide: QCDSM

The example of ρ meson,

- It focuses on the vacuum to vacuum correlation function of the gauge invariant operators in terms of operator product expansions(OPE) [4].

$$\langle 0 \left| T \left\{ \mathcal{O}_1(x) \mathcal{O}_2(0) \right\} \right| 0 \rangle \tag{3}$$

- The ρ meson field representation,

$$\rho_\mu(x) = \bar{u}(x)\gamma_\mu u(x) \; - \; \bar{d}(x)\gamma_\mu d(x)$$

  We built operators out of quark-antiquark fields, then study the behavior of the operators in QCD.

- In general quantum field theory, for scalar field theory, the Fourier transform of the correlation function

$$\int \langle \, T(\phi(x)\phi(0)) \, \rangle \, e^{ip\cdot x} \, dx = \frac{i}{p^2 - m^2}$$

- For ρ meson,

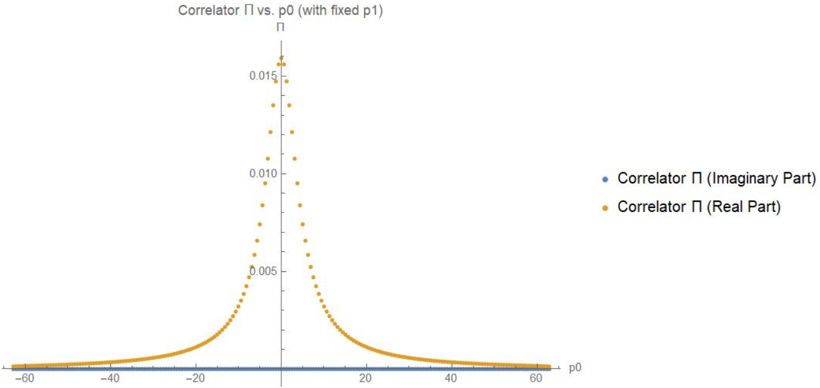
$$\int \langle \, T(\rho_\mu(x)\rho_\nu(0)) \, \rangle \, e^{ip\cdot x} \, dx = \frac{i}{p^2 - m_\rho^2}$$

QCD Duality

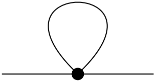
- Another property from QCD theory is called duality.
- If integrate over a wide enough range of energy region, QCD prediction agrees with the results of hadronic physics.
- Imposing duality by equating QCD expression of intergal over all space and hadronic expression of the same integral, we have the relation between the calculatable the LHS (using quark/gluon properties and Feynman diagram) and RHS which contains hadron masses.

$$\oint \frac{\Pi^{QCD}(z)dz}{z-Q^2} = \oint \frac{\Pi^{hadrons}(z)dz}{z-Q^2} \tag{4}$$

pySecDec Numerical Calculation: 1+1 Dimension



The Scalar One-loop Integral



$$
\begin{aligned}
&= \frac{\lambda}{2}\, T \sum_{P} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + \boldsymbol{p}^2 + m^2} \\
&= \frac{\lambda}{2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{P}}} \int_0^{\beta} d\tau \sum_{n} \delta(\tau - n\beta) \Big[(1 + n_B(E_{\boldsymbol{P}}))e^{-E_{\boldsymbol{P}}\tau} + n_B(E_{\boldsymbol{P}})e^{E_{\boldsymbol{P}}} \\
&= \frac{\lambda}{2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{P}}} [1 + 2n_B(E_{\boldsymbol{P}})] \\
&= \lambda \left[\frac{\Lambda^2}{16\pi^2} + \frac{T^2}{24} + \dots \right]
\end{aligned}
$$

Backup Slide: Conversion Factor

A scalar Feynman graph G in d dimensions at one loop with N propagators, where the propagators can have powers of ν_j, has the momentum space representation of

$$G = \int d^d\kappa \, \frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}(\{k\},\{q\},m_j{}^2)}, \tag{5}$$

$$d^d\kappa = \frac{\mu^{4-d}}{i\pi^{\frac{d}{2}}} d^dk, \;\; P_j^{\nu_j} = ((k-q_j)^2 - m_j{}^2 + i\delta),$$

where μ denotes the renormalization scale, the q_j are external momenta, and k is the loop momentum [5]. Whereas the convention of [6] has the integral expression of

$$G = \lim_{\delta\to 0^+} \frac{1}{\mu^{d-4}} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - m^2 + i\delta]^n}. \tag{6}$$

Equation (5) and (6) give a coefficient difference of

$$\frac{1}{i\pi^{\frac{d}{2}}} \times \left[\frac{1}{(2\pi)^d}\right]^{-1} = (2\pi)^d \left(-\frac{i}{\pi^{d/2}}\right) \tag{7}$$

where $d = 4 + 2\epsilon$.

Backup Slide: TBI Integral Expression

The analytical result of the loop integral with both propagator masses are zero is referred from [7] and simplified in [8],

TBI $[\ 4{+}2\epsilon, q^2, \{\nu_1, 0\}, \{\nu_2, 0\}\ ]$

$$= \frac{i}{(4\pi)^2} \left[-\frac{q^2}{4\pi}\right]^{\epsilon} (q^2)^{2-\nu_1-\nu_2} \frac{\Gamma\,[2-\nu_1+\epsilon]\,\Gamma\,[2-\nu_2+\epsilon]\,\Gamma\,[\nu_1+\nu_2-2-\epsilon]}{\Gamma\,[\nu_1]\,\Gamma\,[\nu_2]\,\Gamma\,[4-\nu_1-\nu_2+2\epsilon]},$$

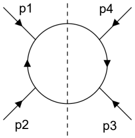
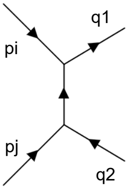
The loop integral for massive TBI in terms of the Gauss Hypergeometric function referring to [7, 8]

$$\text{TBI}\ [\ 4+2\epsilon, q^2, \{\nu_1, m\},\{\nu_2, 0\}\ ] = \frac{i}{(4\pi)^2}\, [-\frac{q^2}{4\pi}]^{\frac{d}{2}-2}\, (q^2)^{2-\nu_1-\nu_2}\ z^{\nu_1+\nu_2-\frac{d}{2}}$$

$$\frac{\Gamma\,[\frac{d}{2}-\nu_1]\,\Gamma\,[\nu_1+\nu_2-\frac{d}{2}]}{\Gamma\,[\nu_1]\,\Gamma\,[\frac{d}{2}]}\, {}_2F_1\,[\nu_1+\nu_2-\frac{d}{2}, \frac{d}{2}-\nu_1; \frac{d}{2}; z]$$

$$\text{TBI}\ [\ 4+2\epsilon, q^2, \{\nu_1, m\},\{\nu_2, m\}\ ] = \frac{i}{(4\pi)^2} \left[-\frac{m^2}{4\pi}\right]^{\frac{d}{2}-2} (-m^2)^{2-\nu_1-\nu_2}$$

$$\times \frac{\Gamma\,[\nu_1+\nu_2-\frac{d}{2}]}{\Gamma\,[\nu_1+\nu_2]}\ 32\ \nu_1\ ,\ \nu_2\ ,\ \nu_1+\nu_2-\frac{d}{2}\,\frac{1}{2}\,(\nu_1+\nu_2)\ ,\ \frac{1}{2}\,(\nu_1+\nu$$

$2\,\mathrm{Im}\Big($  $\Big) = \int d\Pi\,\Big|$  $\Big|^2$ (8)

With $p_1^2 = p_2^2 = p_3^2 = p_4^2$ and $p_4 = p_1 + p_2 + p_3$, we can easily derive

$$-\,p_i \cdot p_j = \frac{p_1^2}{3} = \frac{p_2^2}{3} = \frac{p_3^2}{3} = \frac{q^2}{3}$$

where $i, j = 1, 2, 3, i \neq j$. From the right hand side of eq. (8), we have

$$p_i + p_j = q_1 + q_2$$
$$(p_i + p_j)^2 = (q_1 + q_2)^2$$
$$p_i^2 + p_j^2 + 2p_i \cdot p_j = q_1^2 + q_2^2 + 2q_1 \cdot q_2$$

Backup Slide: Cutting Rules Derivation

As all the internal lines have the same mass m,

$$2\,q^2 - \frac{2}{3}\,q^2 = 2m^2 + 2\,q_1 \cdot q_2.$$

Since $q_1 \cdot q_2 = E_1 E_2 - \boldsymbol{q_1} \cdot \boldsymbol{q_2} > 0$, the relationship between q^2 and mass is then

$$\frac{4}{3}\,q^2 > 2m^2$$

$$q^2 > \frac{3}{2}\,m^2\,.$$