Numerical Loop-Integration Methods for Finite Temperature Effects in QCD Sum Rules

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Thermal Field Theory

- A extension of ordinary Quantum Field Theory at zero temperature.
- There are interesting physical problems that are depart from this situation.
- For example, particles in the early universe, the electroweak phase transition [1], the quark-gluon plasma, gravitational wave signals from strong first-order phase transitions [2].
The contour of the density operator is redefined analogous to the approach of Feynman rules at zero temperature is

\[ C = [t_i, +\infty] \cup [+\infty, t_i] \cup [t_i, t_i - i\beta]. \] [3]

**Figure:** The integration contour \( C \) of thermal time for the density operator \( e^{-\rho H} \) in the complex time plane.
The Feynman diagram integrand can be Fourier transformed into frequency space. For free bosonic scalar fields,

\[ G^0(\omega_n, p) = \frac{1}{\omega_n^2 + p^2 + m^2} \]

where the Matsubara frequencies are \( \omega_n = 2\pi n T \) for \( T \) being temperature. [3]
pySecDec Numerical Calculation

- pySecDec: a package designed for numerical calculation of dimensionally regulated loop integrals.
- The Feynman diagram topology representing the one-loop self-energy integral. [3]

\[
\Pi(p, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\frac{4n^2\pi^2}{\beta^2} + k^2 + M^2} \times \frac{1}{\left(\frac{2n\pi}{\beta} + p^0\right)^2 + (k + p)^2 + M^2}
\]
\[ \Pi(p, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + m_1^2} \times \frac{1}{(k + p)^2 + m_2^2} \]

where we take \( m_1^2 = \frac{4n^2\pi^2}{\beta^2} + M^2 \) and \( m_2^2 = (\frac{2n\pi}{\beta} + p^0)^2 + M^2 \).

- Using pySecDec to do the d-dimensional spatial integration part, then write separate python codes for the discretized series of temporal sum.
- Momentum in correlation function is in Euclidean space whereas pySecDec works on loop integrals in Minkowski space.
- We need to do inverse Euclideanization to the input momentum(\( p_1 \)) by applying Wick rotation.
Methodology for pySecDec Numerical Calculation

\[ p_0^2 - p_1^2 - p_2^2 \quad \text{(1)} \]

(pySecDec)

\[ \downarrow \]

\[ -p_1^2 - p_2^2 - p_3^2 \quad \text{(2)} \]

(what we want for pySecDec to do is pure spatial integration)
\[ \Pi(p_1, p^0) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{dk_1}{2\pi} \frac{1}{4n^2\pi^2 + k_1^2 + M^2} \times \frac{1}{(\frac{2n\pi}{\beta} + p^0)^2 + (k_1 + p_1)^2 + M^2} \]

where we take \( m_1^2 = \frac{4n^2\pi^2}{\beta^2} + M^2 \) and \( m_2^2 = (\frac{2n\pi}{\beta} + p^0)^2 + M^2 \).
Correlator $\Pi$ vs. $p_1$ (with fixed $p_0$)
Correlator vs. $p_1$ with discrete $p_0$ value ($p_0 = \frac{2\pi l}{\beta}, M=1.1\text{GeV}, \beta=10k^{-1}$)

Correlator $\Pi$ (real part)
Finite Temperature Correction vs Temperature (1+1D, M=1.1GeV, $\rho_0 = \frac{8\pi}{\beta}$, $p_1 = 5$)

Finite Temperature Correction ($\varepsilon^0$)

- Real Part
- Imaginary Part
Regulated Correlator (1+3D) vs maximum n of series

- Finite Temperature ($\beta=0.3 K^{-1}$)
- Zero Temperature
- Finite Temperature Correction (Finite–Zero)
We have developed a numerical calculation method for Thermal Field Theory focusing on the one-loop self-energy topology that appears in QCD Sum Rules.

Our methodology incorporates pySecDec for calculating part of the Thermal Field Theory integration and additional coding for the discretized dimension.

There are still many problems and questions emerging in calculations from a variety of dimensions.
Thank You!


The example of $\rho$ meson,

- It focuses on the vacuum to vacuum correlation function of the gauge invariant operators in terms of operator product expansions (OPE) [4].

$$\langle 0 | T \{\mathcal{O}_1(x)\mathcal{O}_2(0)\} | 0 \rangle$$

(3)

- The $\rho$ meson field representation,

$$\rho_\mu(x) = \bar{u}(x)\gamma_\mu u(x) - \bar{d}(x)\gamma_\mu d(x)$$

We built operators out of quark-antiquark fields, then study the behavior of the operators in QCD.

- In general quantum field theory, for scalar field theory, the Fourier transform of the correlation function

$$\int \langle T(\phi(x)\phi(0)) \rangle e^{ip \cdot x} \, dx = \frac{i}{p^2 - m^2}$$

- For $\rho$ meson,

$$\int \langle T(\rho_\mu(x)\rho_\nu(0)) \rangle e^{ip \cdot x} \, dx = \frac{i}{p^2 - m_\rho^2}$$
Another property from QCD theory is called duality.

If integrate over a wide enough range of energy region, QCD prediction agrees with the results of hadronic physics.

Imposing duality by equating QCD expression of integral over all space and hadronic expression of the same integral, we have the relation between the calculatable the LHS (using quark/gluon properties and Feynman diagram) and RHS which contains hadron masses.

\[
\int \frac{\Pi^{QCD}(z)dz}{z - Q^2} = \int \frac{\Pi^{hadrons}(z)dz}{z - Q^2}
\]  

(4)
pySecDec Numerical Calculation: 1+1 Dimension

Correlator $\Pi$ vs. $p_0$ (with fixed $p_1$)

- Blue dots: Correlator $\Pi$ (Imaginary Part)
- Orange dots: Correlator $\Pi$ (Real Part)
The Scalar One-loop Integral

\[
\begin{align*}
&= \frac{\lambda}{2} T \sum_{P} \int \frac{d^{3} p}{(2\pi)^{3}} \frac{1}{\omega_{n}^{2} + p^{2} + m^{2}} \\
&= \frac{\lambda}{2} \int \frac{d^{3} p}{(2\pi)^{3} 2E_{P}} \int_{0}^{\beta} d\tau \sum_{n} \delta(\tau - n\beta) \left[ (1 + n_{B}(E_{P})) e^{-E_{P}\tau} + n_{B}(E_{P}) e^{E_{P}\tau} \right] \\
&= \frac{\lambda}{2} \int \frac{d^{3} p}{(2\pi)^{3} 2E_{P}} \left[ 1 + 2n_{B}(E_{P}) \right] \\
&= \lambda \left[ \frac{\Lambda^{2}}{16\pi^{2}} + \frac{T^{2}}{24} + \ldots \right]
\end{align*}
\]
A scalar Feynman graph $G$ in $d$ dimensions at one loop with $N$ propagators, where the propagators can have powers of $\nu_j$, has the momentum space representation of

$$G = \int d^d\kappa \frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}(\{k\}, \{q\}, m_j^2)},$$

$$d^d\kappa = \frac{\mu^{4-d}}{i\pi^{d/2}} d^d k, \quad P_j^{\nu_j} = ((k - q_j)^2 - m_j^2 + i\delta),$$

where $\mu$ denotes the renormalization scale, the $q_j$ are external momenta, and $k$ is the loop momentum [5]. Whereas the convention of [6] has the integral expression of

$$G = \lim_{\delta \to 0^+} \frac{1}{\mu^{d-4}} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - m^2 + i\delta]^n}.$$  

Equation (5) and (6) give a coefficient difference of

$$\frac{1}{i\pi^{d/2}} \times \left[ \frac{1}{(2\pi)^d} \right]^{-1} = (2\pi)^d \left( -\frac{i}{\pi^{d/2}} \right)$$

where $d = 4 + 2\epsilon$. 
The analytical result of the loop integral with both propagator masses are zero is referred from [7] and simplified in [8],

\[
\text{TBI} \left[ 4+2\epsilon, q^2, \{\nu_1, 0\}, \{\nu_2, 0\} \right] = \frac{i}{(4\pi)^2} \left[ -\frac{q^2}{4\pi} \right]^\epsilon (q^2)^{2-\nu_1-\nu_2} \frac{\Gamma [2 - \nu_1 + \epsilon] \Gamma [2 - \nu_2 + \epsilon] \Gamma [\nu_1 + \nu_2 - 2 - \epsilon]}{\Gamma [\nu_1] \Gamma [\nu_2] \Gamma [4 - \nu_1 - \nu_2 + 2\epsilon]},
\]

The loop integral for massive TBI in terms of the Gauss Hypergeometric function referring to [7, 8]

\[
\text{TBI} \left[ 4 + 2\epsilon, q^2, \{\nu_1, m\}, \{\nu_2, 0\} \right] = \frac{i}{(4\pi)^2} \left[ -\frac{q^2}{4\pi} \right]^\frac{d}{2} - 2 (q^2)^{2-\nu_1-\nu_2} \frac{\Gamma [\frac{d}{2} - \nu_1] \Gamma [\nu_1 + \nu_2 - \frac{d}{2}]}{\Gamma [\nu_1] \Gamma [\frac{d}{2}]} _2F_1 [\nu_1 + \nu_2 - \frac{d}{2}, \frac{d}{2} - \nu_1; \frac{d}{2}; z]
\]

\[
\text{TBI} \left[ 4 + 2\epsilon, q^2, \{\nu_1, m\}, \{\nu_2, m\} \right] = \frac{i}{(4\pi)^2} \left[ -\frac{m^2}{4\pi} \right]^\frac{d}{2} - 2 \frac{\Gamma [\nu_1 + \nu_2 - \frac{d}{2}]}{\Gamma [\nu_1 + \nu_2]} 32 \nu_1 , \nu_2 , \nu_1 + \nu_2 - \frac{d}{2} 2 \frac{1}{2} (\nu_1 + \nu_2), \frac{1}{2} (\nu_1 + \nu_2)
\]
With \( p_1^2 = p_2^2 = p_3^2 = p_4^2 \) and \( p_4 = p_1 + p_2 + p_3 \), we can easily derive

\[
- p_i \cdot p_j = \frac{p_1^2}{3} = \frac{p_2^2}{3} = \frac{p_3^2}{3} = \frac{q^2}{3}
\]

where \( i, j = 1, 2, 3, i \neq j \). From the right hand side of eq. (8), we have

\[
(p_i + p_j)^2 = (q_1 + q_2)^2
\]

\[
p_i^2 + p_j^2 + 2p_i \cdot p_j = q_1^2 + q_2^2 + 2q_1 \cdot q_2
\]
As all the internal lines have the same mass $m$,

$$2 q^2 - \frac{2}{3} q^2 = 2m^2 + 2 q_1 \cdot q_2.$$ 

Since $q_1 \cdot q_2 = E_1 E_2 - q_1 \cdot q_2 > 0$, the relationship between $q^2$ and mass is then

$$\frac{4}{3} q^2 > 2m^2$$

$$q^2 > \frac{3}{2} m^2.$$